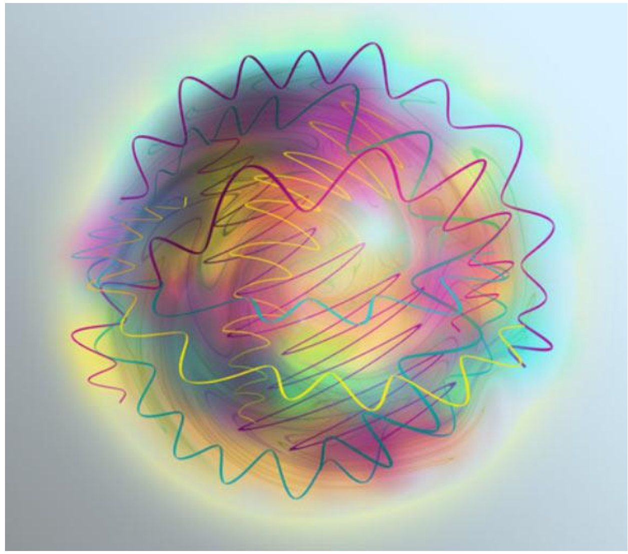
An exceptional *G*(2) extension of the Standard Model from the correspondence with Cayley–Dickson algebras automorphism groups



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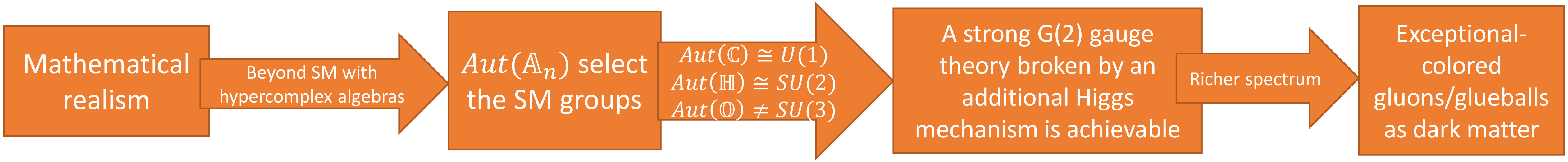
Summary

1. Motivation and preliminaries

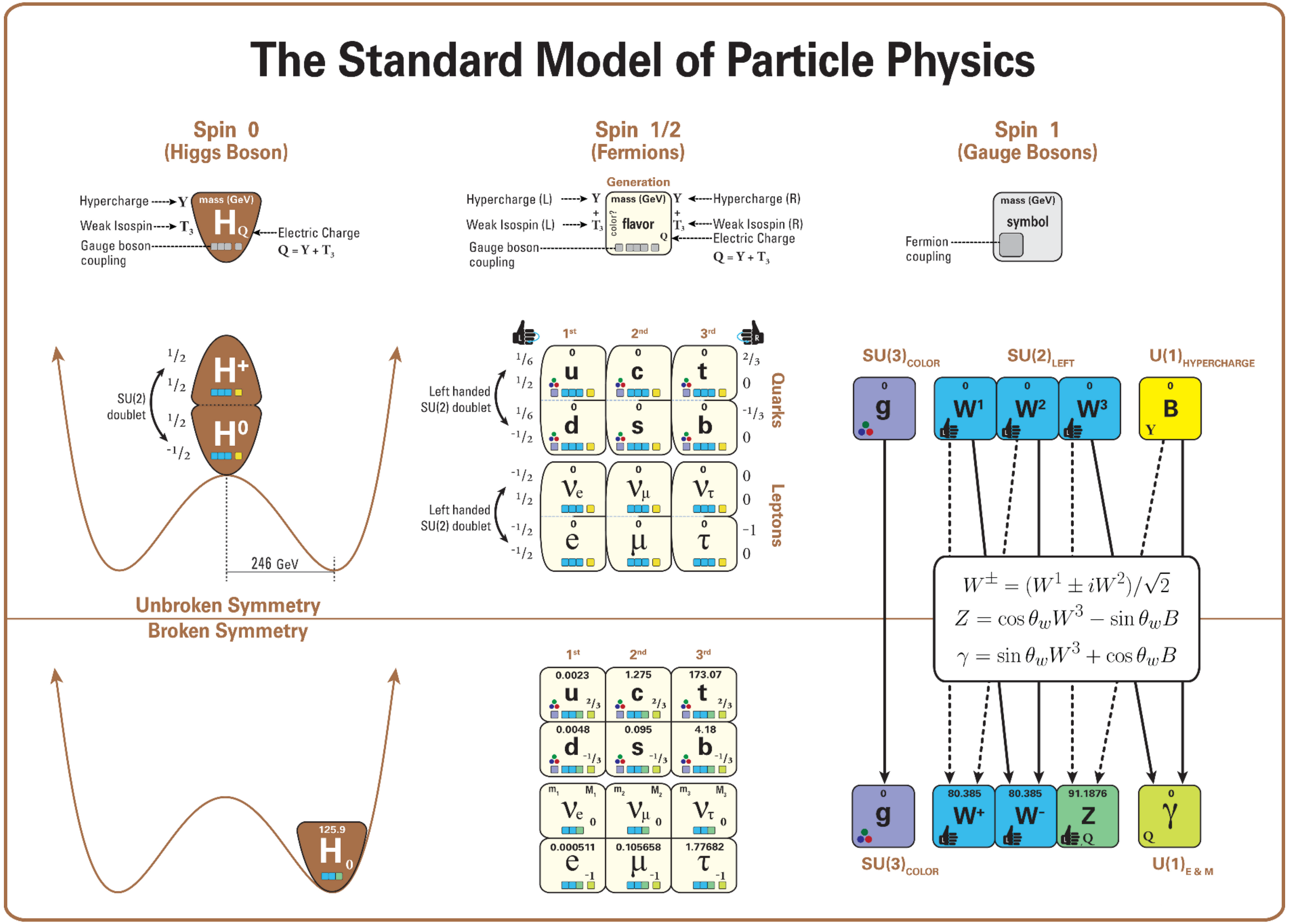
2. The algebraic conjecture (guideline): the automorphism search

3. Developing a broken G(2) gauge theory for the strong sector

4. Describing the exotic bosonic content



1. The Standard Model: the pattern



3 fermions families

$$G = SU(3) \times SU(2) \times U(1)$$

Symmetry breaking

Internal and external problems:

- theoretical issues (hierarchy, strong CP problems, etc…)
- fundamental not explained phenomena, like **dark matter** or dark energy

Naturalness and WIMP Miracle

The fractional abundance, relative to the critical density, and the **thermal averaged annihilation cross section** are

$$\Omega_{WIMP}h^2 \approx 0.1\left(\frac{3\cdot 10^{-26}cm^3s^{-1}}{\langle\sigma v\rangle}\right) = 0.1199 \qquad \langle\sigma v\rangle \simeq \frac{g_X^4}{16\pi^2 m_{DM}^2}(1\ or\ v^2)$$

$$m_{DM}{\sim}100\ GeV, \qquad g_X{\sim}0.6$$

We believed that:

$\langle\sigma v\rangle \approx 3\cdot 10^{-26}cm^3s^{-1}$ is a typical value expected for a particle with mass near the weak scale [O(100 GeV)] and a weak gauge coupling: the fact that the observed abundance of DM points to new physics at the weak scale, independently of particle physics motivations, was the so-called **WIMP miracle** (a **Naturalness Criterion**).

But

- Any combination of m_{DM}, g_X can be taken.
- Griest and Kamionkowski applied the **Unitarity Bound** to infer an upper limit on the dark matter particle mass: using PLANCK constraints, this bound is something like: **$m_{DM} \lesssim$ 120 TeV** (for a scalar). It slightly varies according to the spin-statistics of the candidate. A recent study shows a **$m_{DM} \lesssim$ 139 TeV** constraint for a Dirac fermion.
- LHC searches **excluded most of the WIMP DM candidates up to 600 GeV-1 TeV scale** and we have **no evidence from direct and indirect searches**

Searching for a mathematical guidelines: Symmetries of Cayley-Dickson algebras

Cayley-Dickson construction: one can build up a **sequence of larger and larger algebras, adding new imaginary units**. During the construction process, the algebras lose some peculiar properties, one at a time: complex numbers are not ordered but commutative, quaternions are not commutative but associative, whereas octonions lose all the familiar commutative and associative properties.

Hurwitz and Zorn theorem: one can identify the so-called division algebras $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$, i.e. the **only four alternative algebraic fields with no non-trivial zero divisors** - non-zero element *a* of a ring *R* is called a zero divisor if there exists a non-zero *x* such that $ax = 0$ – which give problems in the definition of norm.

Automorphism: an isomorphism from a mathematical object to itself, a way of mapping the object while preserving all of its structure. **The set of all automorphisms is the automorphism group, i.e. the symmetry group of the object**. Unlike Lie algebras and Clifford algebras, there is a finite number of division algebras and corresponding automorphisms.

2. Complex numbers symmetry → $U(1)$

- The existence of infinite distinct ***wild automorphisms*** of the complex numbers, beyond identity and complex conjugation, is well-known

- The group **U(1), the smallest compact real Lie group**, corresponds to the circle group S^1, consisting of all complex numbers with absolute value 1 under multiplication, which is isomorphic to the SO(2) group of rotation:

$$n\text{-torus } T_n = \mathbb{R}^n/\mathbb{Z}^n \cong U(n) \cong SO(2)^n \cong (S^1)^n$$

- The n × n complex matrices which leave the scalar product invariant form the group $U(n) = Aut(\mathbb{C}^n, \langle , \rangle)$

- U(1) numbers effectively operate as automorphisms of $\mathbb{C}$ via multiplication of a phase factor:

$$\mathbb{C}^* = \mathbb{C}\backslash 0 = e^{\mathbb{C}} \cong GL(1,\mathbb{C})$$

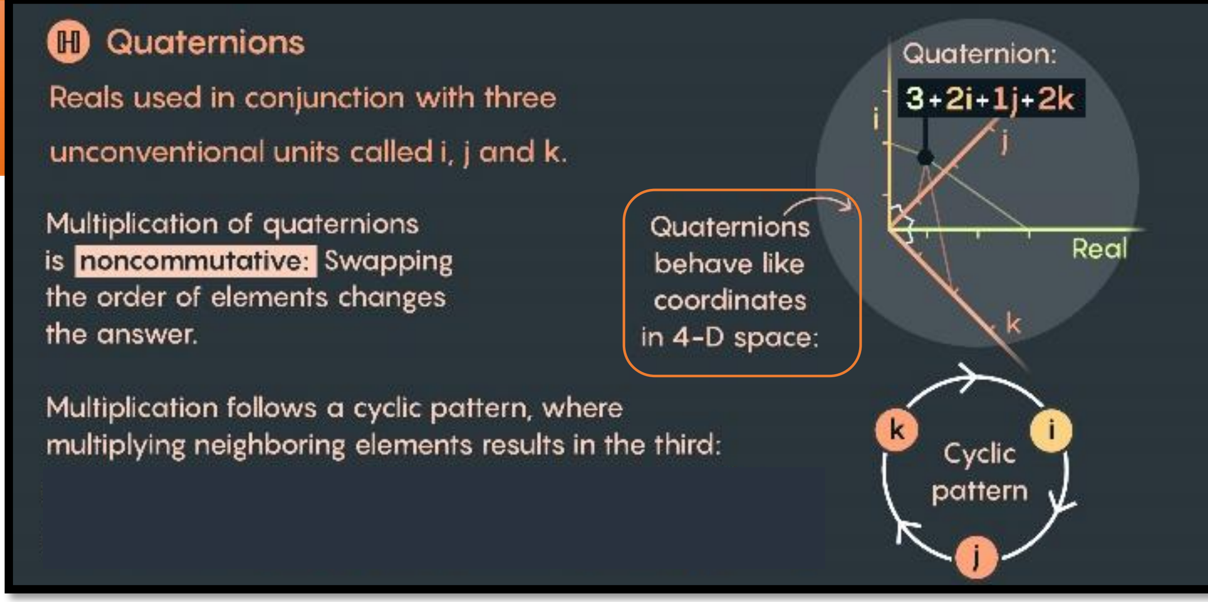
→ $$|\mathbb{C}^*| = e^{\mathbb{R}}$$

→ $$\mathbb{C}^*/|\mathbb{C}^*| = e^{i\mathbb{R}} \cong U(1)$$

- We know $\mathbb{C}$ can be applied to many aspects of real life, especially in electronics and electromagnetism: **Riemann-Silberstein field** reformulation of the electromagnetism in terms of a complex vector that combines the electric field E, as the real part, and the magnetic field B, as the imaginary part, can put in evidence this essential relation:

$$\mathbf{F} = \mathbf{E} + ic\mathbf{B}$$

Quaternions → $SU(2)$



- Quaternions were first described by Irish mathematician William Rowan Hamilton in 1843 and applied to mechanics in three-dimensional space
- There is a strong relation between quaternion units and Pauli matrices:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin/isospin group language $\mathfrak{su}(2) = \mathrm{span}\{\, i\,\sigma_1 \,,\, i\,\sigma_2 \,,\, i\,\sigma_3 \,\}$

Quaternionic basis $\mathbf{1} \mapsto I, \quad \mathbf{i} \mapsto -\sigma_2\sigma_3 = -i\,\sigma_1, \quad \mathbf{j} \mapsto -\sigma_3\sigma_1 = -i\,\sigma_2, \quad \mathbf{k} \mapsto -\sigma_1\sigma_2 = -i\,\sigma_3$

- The Lie algebras **so(3) and su(2) are isomorphic** and $Aut(\mathbb{H}) = SO(3)$, where *SO*(3) is homomorphic to *SU*(2), and the universal cover of *SO*(3) is the spin group *Spin*(3), which is isomorphic to *SU*(2)
- The quaternionic representation of (electro-)weak isospin has been used by many authors in literature.

Cayley numbers: largest non-associative division algebra

Triads

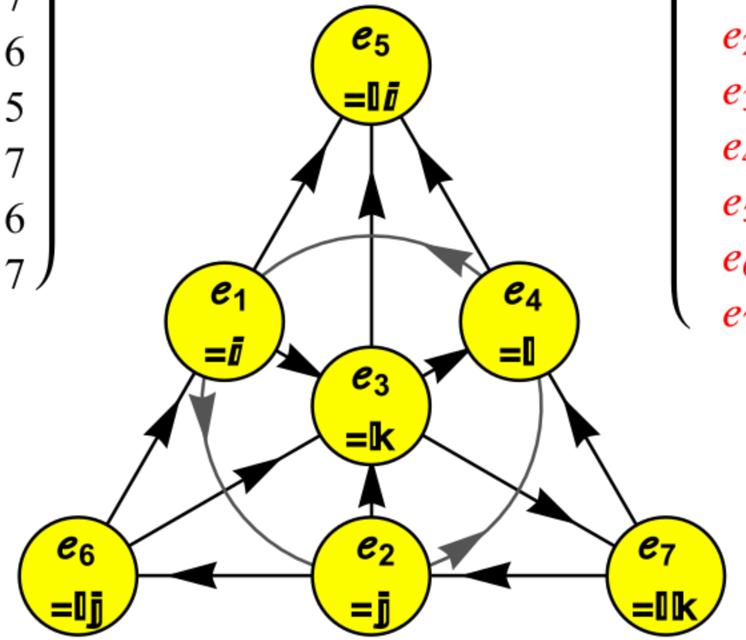
$$\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 5 & 6 \\ 2 & 3 & 5 \\ 2 & 6 & 7 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \end{pmatrix}$$

$e_1 e_2 = e_4$

$e_1 e_3 = e_7$

....

Octonion Fano Plane



e_n

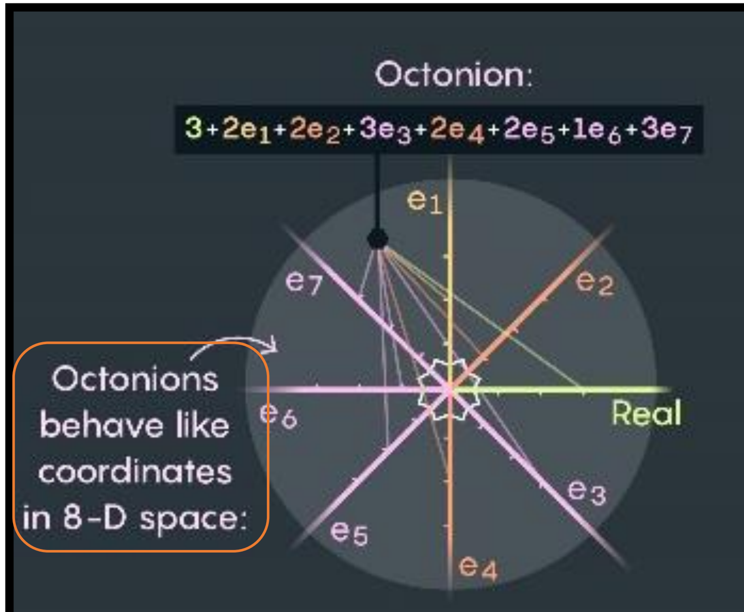
$e_0 \mapsto 1$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

IJKL

$e_0 \mapsto 1$	𝕚	𝕛	𝕜	𝕝	𝕝𝕚	𝕝𝕛	𝕝𝕜
𝕚	−1	𝕝	𝕝𝕜	−𝕛	𝕝𝕛	−𝕝𝕚	−𝕜
𝕛	−𝕝	−1	𝕝𝕚	𝕚	−𝕜	𝕝𝕜	−𝕝𝕛
𝕜	−𝕝𝕜	−𝕝𝕚	−1	𝕝𝕛	𝕛	−𝕝	𝕚
𝕝	𝕛	−𝕚	−𝕝𝕛	−1	𝕝𝕜	𝕜	−𝕝𝕚
𝕝𝕚	−𝕝𝕛	𝕜	−𝕛	−𝕝𝕜	−1	𝕚	𝕝
𝕝𝕛	𝕝𝕚	−𝕝𝕜	𝕝	−𝕜	−𝕚	−1	𝕛
𝕝𝕜	𝕜	𝕝𝕛	−𝕚	𝕝𝕚	−𝕝	−𝕛	−1

- It is possible to **fix one of the octonion basis elements to obtain seven possible subalgebras**, each of which has a subgroup of automorphisms isomorphic to SU(3)

- Recent works show the possibility to **rewrite *Gell-Mann matrices* of *SU*(3) strong force with octonions**. Also *split-octonions* representations have been proposed as alternative formalism for *SU*(3) color gauge symmetry



The automorphism discrepancy and the emergency of the exceptional groups

$$\mathrm{Aut}(\mathbb{O}) = G(2)$$

The Standard Model gauge group SU(3) is not isomorphic to the group of automorphisms of the octonions G(2)

- The exceptional **G(2) group is certainly much bigger than Standard Model SU(3)**, as it includes SU(3) and is equipped with six additional generators
- **Furey** has recently suggested the appealing possibility to reformulate the SM group G = SU(3) × SU(2) × U(1) in terms of a $\mathbb{A} = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ tensor product algebra, restarting from **Dixon**'s work, using the concept of *Ideals*, i.e. using subspaces of proper Clifford Algebras as "particles"
- Whereas $\mathbb{R}$,$\mathbb{C}$,$\mathbb{H}$ and $\mathbb{O}$ are by themselves division algebras, their tensor products, such as $\mathbb{C} \otimes \mathbb{H}$, $\mathbb{H} \otimes \mathbb{O}$ and $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, **largely applied in SM algebraic extensions**, are not:

Freudenthal–Tits magic square

$\otimes$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$Sp(3)$	F_4
$\mathbb{C}$	$SU(3)$	$SU(3)^2$	$SU(6)$	E_6
$\mathbb{H}$	$Sp(3)$	$SU(6)$	$SO(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Let's go further

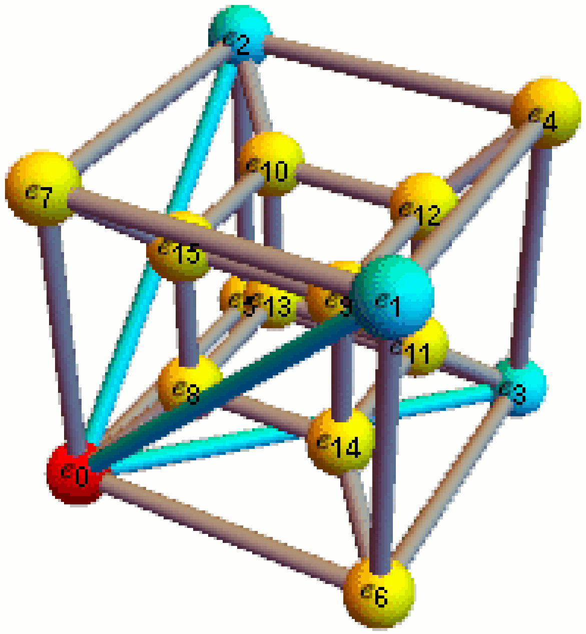
Sedenions: the archetype of generalized non division algebras

$$E_{16} = \{e_i \in \mathbb{S} | i = 0, 1, \dots, 15\}$$

$$A = \sum_{i=0}^{15} a_i e_i = a_0 + \sum_{i=1}^{15} a_i e_i, \qquad a_i \in \mathbb{R}$$

$$\begin{cases} e_0 = 1, & e_0 e_i = e_i e_0 = e_i, \\ e_i^2 = -e_0, & i \neq 0, \\ e_i e_j = \gamma_{ij}^k e_k & i \neq 0, \quad i \neq j \end{cases}$$

$$AB = \left(\sum_{i=0}^{15} a_i e_i\right)\left(\sum_{i=0}^{15} b_j e_i\right) = \sum_{i,j=0}^{15} a_i b_j (e_i e_j) = \sum_{i,j,k=0}^{15} f_{ij} \gamma_{ij}^k e_k \qquad f_{ij} \equiv a_i b_j$$



Because the sedenion algebra is not a division algebra, it contains **zero divisors (84):**

$$(e_a + e_b) \circ (e_c + e_b) = 0, \qquad e_a, e_b, e_c, e_d \in \mathbb{S}$$

The only difference between octonions and sedenions automorphism groups is a **factor of the permutation group S_3**:

$$\mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\mathbb{O}) \times S_3 \quad \Rightarrow \quad \mathrm{Aut}(\mathbb{A}_n) \cong \mathrm{Aut}(\mathbb{O}) \times (n-3) S_3$$

The underlying symmetry is always G(2): **the higher Cayley-Dickson algebras only add additional trialities, i.e. copies of G(2)**, and reasonably no new physics beyond sedenions.
We can stop: sedenions might represent the archetype of all non-associative and nondivision flexible algebras.

The conjecture

$$\mathrm{Aut}(\mathbb{C}) \cong U(1),\ \mathrm{Aut}(\mathbb{H}) \cong SU(2),\ \mathrm{Aut}(\mathbb{O}) \equiv G(2)$$

- The fundamental symmetry of the Standard Model of particle physics with N = 3 fermion families might be the realization of some **tensor products between the associative division algebras and the most comprehensive non-division algebra** obtained through the Cayley–Dickson construction
- Fundamental forces might be **isomorphic to the automorphisms groups**:

EW Strong Families

$$\mathrm{Aut}(\mathbb{C}) \times \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\mathbb{C}) \times \mathrm{Aut}(\mathbb{H}) \times \mathrm{Aut}(\mathbb{O}) \times S_3 \cong \underbrace{U(1) \times SU(2)}_{\text{EW}} \times \underbrace{G(2)}_{\text{Strong}} \times \underbrace{S_3}_{\text{Families}}$$

$(aa)b = a(ab), (ba)a = b(aa)$ $a(ba) = (ab)a$

Charge (n_g)	Group	Force	Algebra	Dim	Commutative	Associative	Alternative	Normed	Flexible
$Q(1)$	$U(1)$	EM	$\mathbb{C}$	2	Yes	Yes	Yes	Yes	Yes
$T(3)$	$SU(2)$	Weak	$\mathbb{H}$	4	No	Yes	Yes	Yes	Yes
$C(8)$	$SU(3)$	Strong	$\mathbb{O}$	8	No	No	Yes	Yes	Yes
$EC(6)$	Broken-$G(2)$	Exceptional strong	$\mathbb{O}$ or $\mathbb{S}$	8/16	No	No	No	No	Yes

- A **new particle content** come from **difference between G(2) and SU(3)** groups and lie in the spectrum gap between them: no more physics is needed nor predicted, except for the six additional degrees of freedom/generators/boson fields
- This is a simple algebraic criterion to **predict physics beyond the Standard Model**, alternative to Higgs Naturalness and the Wimp Miracle.

3. A G(2) gauge theory for the strong sector: why?

The strong force acquires an enlarged **exceptional-colored dynamics**: to recover standard SU(3) color strong force description, the new G(2) color sector should be broken by a Higgs-like mechanism and separated into two parts, one visible and the other excluded from the dynamics due to its peculiar properties.

<u>Compact (semi)simple Lie groups are completely described by the following classes:</u>

$$A_N(= SU(N+1)), B_N(= SO(N+1)), C_N(= Sp(N)), D_N(= SO(2N)) \qquad G_2, F_4, E_6, E_7, E_8$$

1. Only SU(2), SU(3), SO(4) and symplectic Sp(1) have 3-dimensional irreducible representations and **only SU(3) has a complex triplet representation:** one of the historical criteria to associate SU(3) to the three color strong force, with quark states different from antiquarks states
2. There is **only one non-Abelian simple compact Lie algebra of rank 1**: SO(3) ≃ SU(2) = Sp(1), which describes the **weak force** (and the smallest compact real Lie group $T_1 = U(1)$ for EM)
3. There are **four of rank 2**, which generate the groups G(2), SO(5) ≃ Sp(2), SU(3) and SO(4) ≃ SU(2) ⊗ SU(2), with **14, 10**, 8 and 6 generators
4. G(2) is of particular interest because it has a trivial center (the identity) and it is its own universal covering group. It can be also used to mimic **QCD in lattice simulations**, avoiding the so-called **sign problem** (or complex action problem) which afflicts $SU(3)$ and prevents Montecarlo integration.

7x7 real matrices

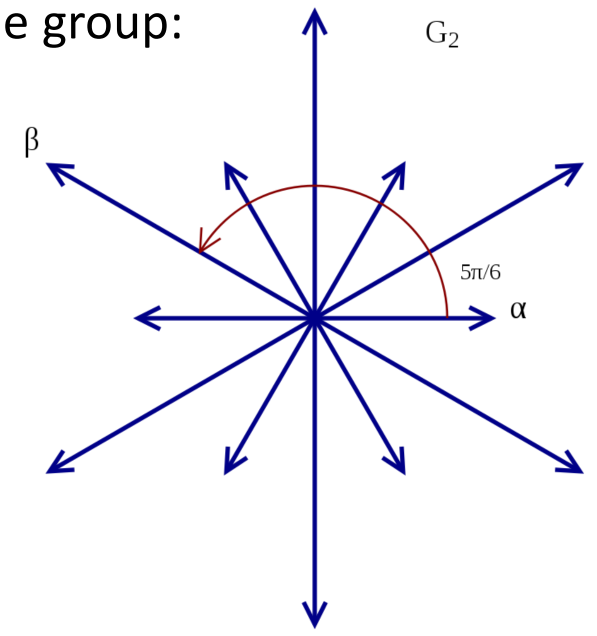
$$\Lambda_a = \frac{1}{\sqrt{2}}\begin{pmatrix} \lambda_a & 0 & 0 \\ 0 & -\lambda_a^* & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>anti-symmetric tensor</u> $T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1$

<u>Gell-Mann-like relations</u> $\mathrm{Tr}\, \lambda_a \lambda_b = \mathrm{Tr}\, \Lambda_a \Lambda_b = 2\delta_{ab}$

Beyond Gell-Mann matrices

Six additional generators can be found studying the root and weight diagrams of the group:



$$\Lambda_9 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & -i\lambda_2 & \sqrt{2}e_3 \\ i\lambda_2 & 0 & \sqrt{2}e_3 \\ \sqrt{2}e_3^T & \sqrt{2}e_3^T & 0 \end{pmatrix}, \Lambda_{10} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & -\lambda_2 & i\sqrt{2}e_3 \\ -\lambda_2 & 0 & -i\sqrt{2}e_3 \\ -i\sqrt{2}e_3^T & i\sqrt{2}e_3^T & 0 \end{pmatrix},$$

$$\Lambda_{11} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & i\lambda_5 & \sqrt{2}e_2 \\ -i\lambda_5 & 0 & \sqrt{2}e_2 \\ \sqrt{2}e_2^T & \sqrt{2}e_2^T & 0 \end{pmatrix}, \Lambda_{12} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & \lambda_5 & i\sqrt{2}e_2 \\ \lambda_5 & 0 & -i\sqrt{2}e_2 \\ -i\sqrt{2}e_2^T & i\sqrt{2}e_2^T & 0 \end{pmatrix},$$

$$\Lambda_{13} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & -i\lambda_7 & \sqrt{2}e_1 \\ i\lambda_7 & 0 & \sqrt{2}e_1 \\ \sqrt{2}e_1^T & \sqrt{2}e_1^T & 0 \end{pmatrix}, \Lambda_{14} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & -\lambda_7 & i\sqrt{2}e_1 \\ -\lambda_7 & 0 & -i\sqrt{2}e_1 \\ -i\sqrt{2}e_1^T & i\sqrt{2}e_1^T & 0 \end{pmatrix},$$

G(2) long roots coincide with SU(3) ones

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

G(2) representation and particle content

Fundamental (for fermions): $\{7\} = \{3\} \oplus \{\bar{3}\} \oplus \{1\}$

Adjoint (for bosons): $\{14\} = \{8\} \oplus \{3\} \oplus \{\bar{3}\}$

- Since all $G(2)$ fundamental representations are real, the {7} representation is identical to its complex conjugate: $G(2)$ "quarks" and "anti-quarks" are conceptually indistinguishable
- $G(2)$ "gluons" ensemble is made of $SU(3)$ gluons {8} plus six additional "gluons" which have $SU(3)$ **quark and anti-quark color quantum numbers**

Products/composite particle content:

$\{7\} \otimes \{7\} = \{1\} \oplus \{7\} \oplus \{14\} \oplus \{27\}$ ⟷ diquarks (J=0,1)

$\{7\} \otimes \{7\} \otimes \{7\} = \{1\} \oplus 4\,\{7\} \oplus 2\,\{14\} \oplus 3\,\{27\} \oplus 2\,\{64\} \oplus \{77\}$ ⟷ baryons

$\{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} = 4\{1\} \oplus \cdots$

$\{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} = 10\{1\} \oplus \cdots$

$\{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} \otimes \{7\} = 35\{1\} \oplus \cdots$

} tetra, penta and exaquarks

$\{7\} \otimes \{14\} \otimes \{14\} \otimes \{14\} = \{1\} \oplus \cdots$ ⟷ q-g hybrids (J=1/2)

$\{14\} \otimes \{14\} = \{1\} \oplus \cdots$

$\{14\} \otimes \{14\} \otimes \{14\} = \{1\} \oplus \cdots$

glueballs

G(2)-Higgs Lagrangian

Yang-Mills Lagrangian: $\mathcal{L}_{YM}[A] = -\frac{1}{2}\,\mathrm{Tr}\,[F_{\mu\nu}^2]$

Field strength: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_G[A_\mu, A_\nu]$

Vector potential: $A_\mu(x) = A_\mu^a(x)\frac{\Lambda_a}{2}$

- G(2) Yang–Mills theory is **asymptotically** free like all non-Abelian SU(N) gauge theories and we expect confinement at low energies
- But the **G(2) confinement is surely peculiar** with a different realization with respect to SU(3), where gluons cannot screen quarks

The Higgs mechanism in action

$$\mathcal{L}_{G_2H}[A,\Phi] = \mathcal{L}_{YM}[A] + (D_\mu\Phi)^2 - V(\Phi)$$

Covariant derivative: $D_\mu\Phi = (\partial_\mu - ig_G A_\mu)\Phi$

Quadratic scalar potential: $V(\Phi) = \lambda(\Phi^2 - w^2)^2$

Real-valued Higgs-like field: $\Phi(x) = (\Phi^1(x), \Phi^2(x), \ldots, \Phi^7(x))$

Vev: $\Phi_0 = \frac{1}{\sqrt{2}}(0,0,0,0,0,0,w)$

$$\begin{cases} \Lambda_{1-8}\Phi_0 = 0 & \text{(unbroken generators)} \\ \Lambda_{9-14}\Phi_0 \neq 0 & \text{(broken generators)} \end{cases}$$

Mass and confinement

Plugging the *vev* into the square of the Higgs covariant derivative we get the usual quadratic term:

Diagonal mass matrix for the gauge bosons:

$$g_G^2 \Phi_0^\dagger \frac{\Lambda_a}{2} \frac{\Lambda_b}{2} \Phi_0 A_\mu^a(x) A^{\mu,b}(x) = \frac{1}{2} \boxed{M_{ab}} A_\mu^a(x) A^{\mu,b}(x)$$

$$M_G = g_G w$$

6 massless Goldstone bosons are eaten and become the longitudinal components of *G*(2) vectorgluons

$$M_H = \sqrt{2\lambda} w$$

1 extra Higgs (from the expansion of the potential about its minimum)

- *G*(2) gauge theory has a **finite-temperature deconfining phase transition mainly of first order**
- **The breaking of the string between two static *G*(2) "quarks" happens due to the production of two triplets of *G*(2) "gluons" which screen the quarks**
- **The larger is M_G, the greater is the distance where string breaking occurs**: when $w \to \infty$, so that the 6 massive *G*(2) "gluons" are completely removed from the dynamics, the scale is infinite and the usual *SU*(3) string potential reappears
- **For small *w* (on the order of Λ_{QCD}) the additional *G*(2) "gluons" could be light** and participate in the dynamics
- If we move away the six G(2) gluons from the dynamics with a high w, these bosons are secluded from the visible SM sector: extreme energies should be mandatory to access the G(2) string breaking. This could be due to the very **high energy scale of the G(2)–SU(3) phase transition**: this could be the **realization of a beyond Naturalness criterion**.

1 new heavy scalar and 6 new vectors for future LHC searches

4. G(2)-Glueballs features

G(2) gluons are electrically neutral and immune to interactions with light and weak W, Z bosons at tree level and no additional families are added to the Standard Model: is this a good scenario for a cold dark matter?

$$\{14\} \otimes \{14\} = \{1\} \oplus \{14\} \oplus \{27\} \oplus \{77\} \oplus \{77'\}$$

$$\{14\} \otimes \{14\} \otimes \{14\} = \{1\} \oplus \{7\} \oplus 5\{14\} \oplus 3\{27\} \oplus 2\{64\} \oplus 4\{77\} \oplus 3\{77'\} \oplus \{182\} \oplus 3\{189\} \oplus \{273\} \oplus 2\{448\}$$

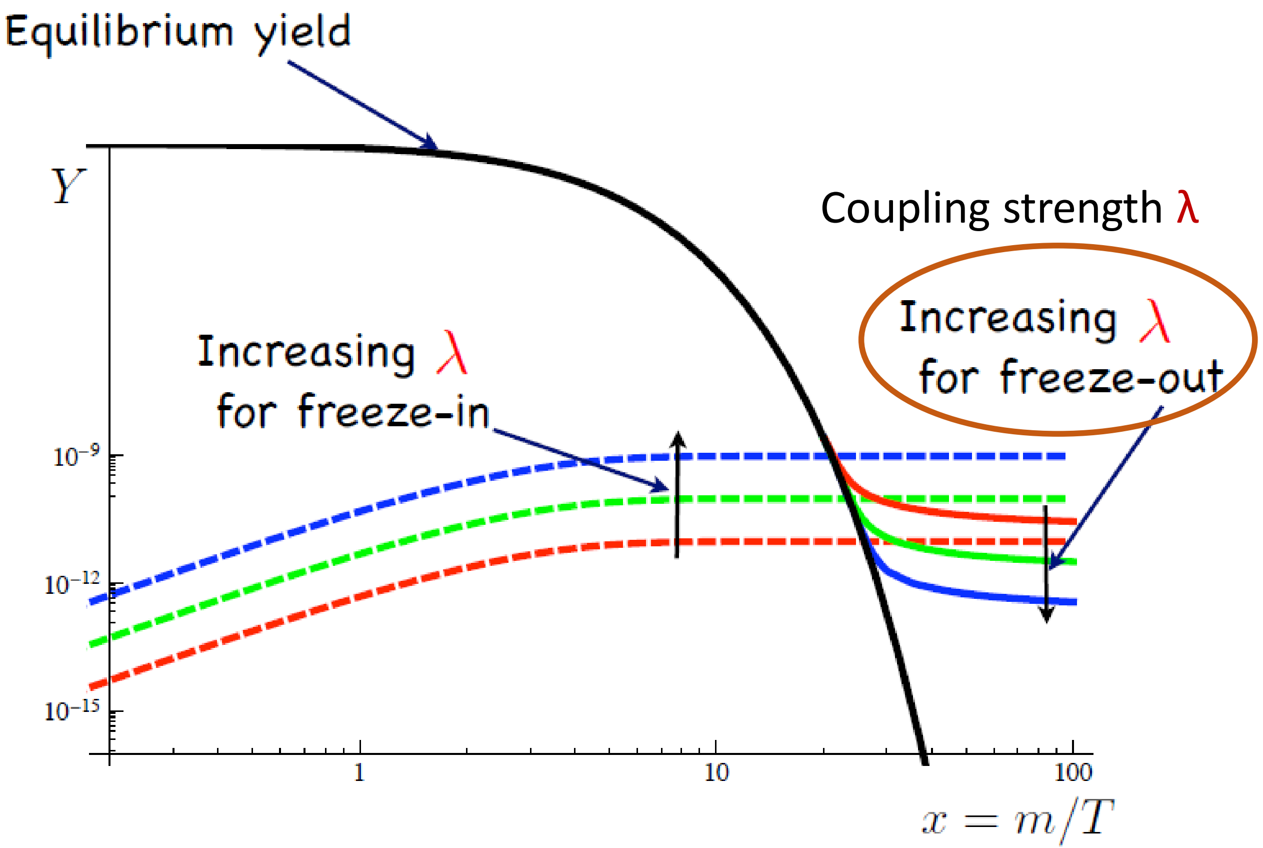
$$\Rightarrow \quad \boxed{J^{PC} = 0^{++}, 2^{++}}$$

There exist **states that couple to both the exceptional-colored glueballs and SU(3) particles** (the G(2)-breaking Higgs field at least!): whether at tree-level or via loops, these heavy glueballs could not be stable…

Conditio sine qua non:

1. $M_H > M_{GG}$
2. *An accidental symmetry: a* conserved additive *gluon number* Γ to prevent the decay into mesons (like *B* for protons)
3. Global discrete Z_2 or generally Z_N symmetry: DM is odd under the new symmetry while SM fields are assumed to be even
4. *G-parity conservation* for a generic Yang–Mills theory, to prevent decay into G-even SM particles (unlike π)

Cosmology: freeze in vs WIMP-Miracle freeze out



$$Y_{FO} \sim \frac{1}{\lambda^2}\left(\frac{m}{M_{Pl}}\right) \qquad Y_{FI} \sim \lambda^2\left(\frac{M_{Pl}}{m}\right)$$

- **FIMP** (Feebly Interacting Massive Particle) cosmology via a freeze-in mechanism: this requires an extremely small coupling ($< 10^{-7}$) with the visible sector

- Another intriguing alternative is the so-called **Dark freeze-out**, for which DM reaches an equilibrium heat bath within the dark sector itself, never interacting with SM particles: the dark ensemble was initially populated by a freeze-in-type yield from part of the visible sector

Scalar G(2)-Glueball cosmology

Given the forbidden or extremely weak interactions between G(2) glueballs and ordinary matter, **the usual WIMP-like scenario built via the freeze-out mechanism cannot be achieved**, since these bosons are never in thermal equilibrium with the baryon-photon fluid in the early Universe: their production should be abruptly triggered by a first-order cosmological phase transition.

Scalar glueball effective SU(N) potential:

$$V(S) = \sum_{i=2}^{\infty} \frac{a_i}{i!}\left(\frac{4\pi}{N}\right)^{i-2} m^{4-i} S^i$$

Exotic-Higgs portal:

$$V(\Phi, S) = V(\Phi) + \frac{m^2}{2}S^2 + \frac{\lambda_S}{4}S^4 + \frac{\lambda_{HS}}{2}\Phi^2 S^2$$

Higgs-Glueball Coupling intesity:

$$\lambda_{HS} \simeq 10^{-12}\left(\frac{\Omega_S h^2}{0.12}\right)^{1/2}\left(\frac{M_H}{M_S}\right)^{1/2}$$

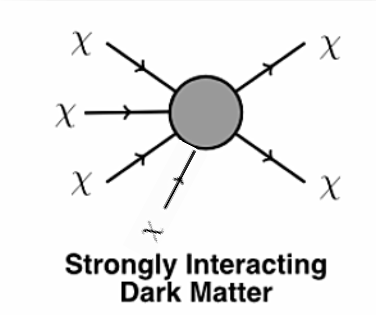
Relic density:

$$M_S = M_{GG} \simeq 2 g_G w \approx \sqrt{\lambda_{HS}/2}\, w \quad \rightarrow \quad \boxed{\Omega_S h^2/0.12 \simeq 10^{24} \lambda_{HS}^{5/2} (w/M_H)}$$

$$w/M_H = O(1) \quad \rightarrow \quad \boxed{\lambda_{HS} \sim 10^{-(10\div9)}}$$

FIMP-like scenario

Self-interaction: $\boxed{\lambda_S > 10^{-3}}$

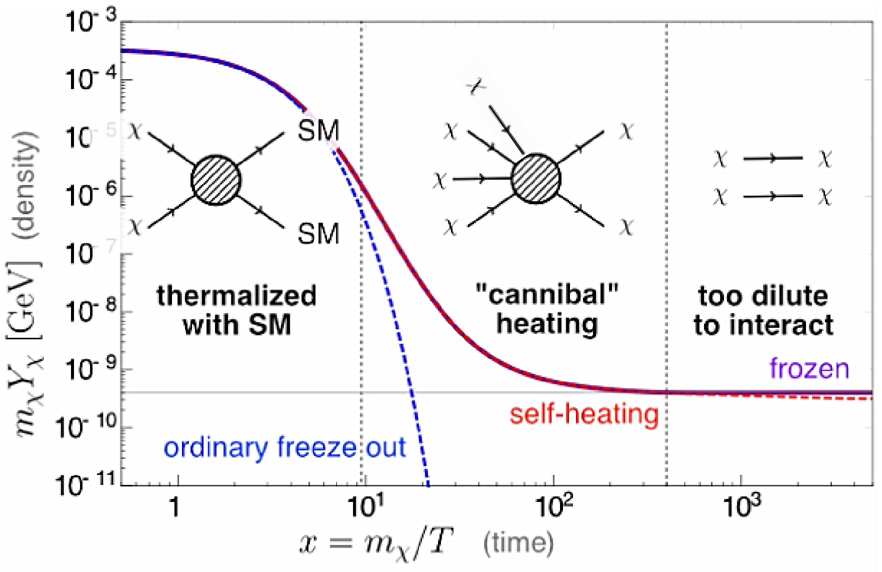


SIMP scenario with Dark freeze-out due to number changing processes

When DM is no longer relativistic, 4 → 2 processes dominate the dynamics in the so-called ***cannibalization era***, which ends when its rate drops below the Hubble rate, fixing the DM number density to a modified yield through the **dark freeze-out**

If DM relic abundance is solely computed via a Higgs decay in a freeze-in framework, without dark thermalization, it could be **appreciably underestimated**

Boltzmann equation for SIMP DM usually leads to scenarios where the **dark freeze-out temperature is less than the visible ensemble one**, making DM naturally colder than SM particles



For even smaller interactions with the visible sector, the thermal production of DM particles is insignificant and DM must come from a non-thermal mechanism, leading to a **Super-WIMP (SWIMP)-like scenario**, for example through a direct DM-producing inflaton decay:

Inflaton-portal

$$\Omega_S h^2 \simeq 2 \times 10^8 B_S (M_S/M_H) \frac{T_{rh}}{\text{GeV}}$$

The scenario works well in literature for large DM masses, between the weak scale and the PeV scale, and for extremely decoupled EeV candidates.

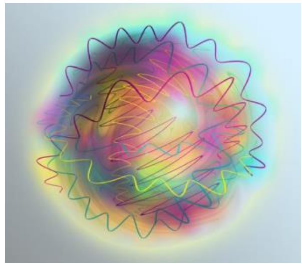
G(2)-glueballs stars

Einstein-Klein-Gordon (EKG) action

$$\mathscr{S} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - g^{\mu\nu}\partial^{\mu}\bar{S}\partial_{\nu}S - V(S) \right]$$

To build up a massive boson star, a repulsive quartic self-interaction potential is needed to balance the inward gravitational force: $\frac{\lambda_4}{4}|S|^4$

Boson stars



$M_{Pl} \sim M_{\odot}^{1/3}$

$$M_{max} \sim (0.1 \div 1) \frac{\sqrt{\lambda_4} M_{Pl}^3}{M_S^2/\text{GeV}^2}$$

$$R_{BS} \sim (0.1 \div 1) \frac{\sqrt{\lambda_4} \times 10\,\text{km}}{M_S^2/\text{GeV}^2}$$

Perturbativity: $\lambda_4 < 4\pi$

$M_S \sim 10$ TeV $\quad M_{max} \lesssim 10^{-8}\sqrt{\lambda_4} M_{\odot}$ and $R_{BS} \lesssim 10^{-4}\sqrt{\lambda_4}$ m

Tiny balls of glueballs

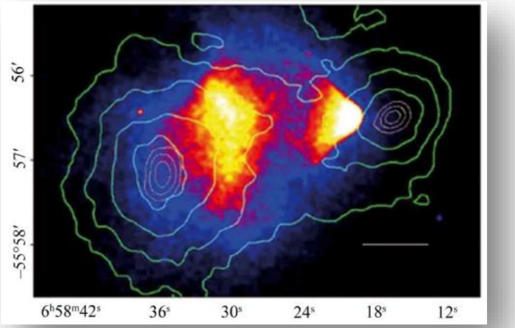
1 GeV scalar $\quad M_{max} \lesssim \sqrt{\lambda_4} M_{\odot}$ and $R_{BS} \lesssim \sqrt{\lambda_4} \times 10$ km

Solar mass objects with a neutron star-like radius

The feasibility of a star made of self-interacting scalars also depends on a **correct estimate of** the possible 3 → 1 , 3 → 2 and 4 → 2 **annihilation processes inside the star**, which in turn depend on the symmetries (Z_N) of the exotic sector

Bullet Cluster self-interaction constraint

$$\frac{\sigma}{M_S} < 1\,\frac{\mathrm{cm}^2}{\mathrm{g}} \sim 10^{-24}\,\frac{\mathrm{cm}^2}{\mathrm{GeV}}$$



$$M_S = M_{GG} \simeq 2g_G w \approx \sqrt{\lambda_{HS}/2}\,w$$

Assuming $\sigma = \frac{9\lambda_4^{\ 2}}{32\pi M_S^2}$ at tree level → $|\lambda_4| < 4 \cdot 10^2 \left(\frac{M_S}{\mathrm{GeV}}\right)^{3/2}$ → ($\lambda_{HS} \sim 10^{-10}$) $|\lambda_4| < 10^{-5} \left(\frac{w}{\mathrm{GeV}}\right)^{3/2}$

Assuming an SU(4)-like coupling $\lambda_4 \sim \pi^2$ → Beyond SM scale → $w \geq 10^4\,\mathrm{GeV}$ → FIMP-SIMP ($w/M_H = O(1)$) → $M_S \gtrsim 10^{-(5\div 4)} M_H$

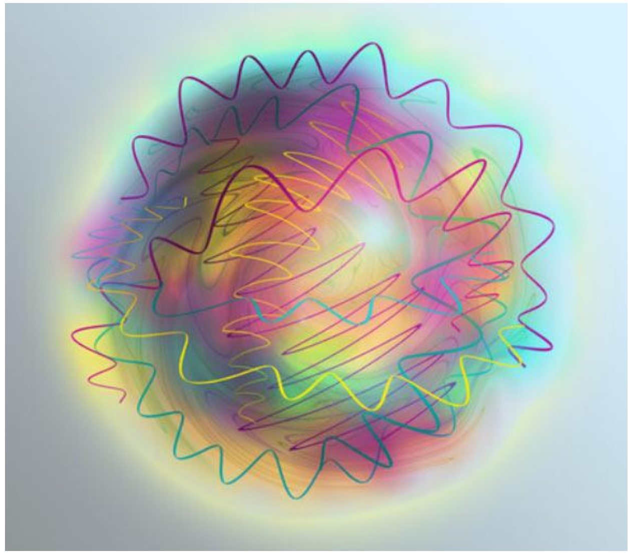
How to probe?

- For a few years we can take advantage of both electromagnetic and **gravitational waves astronomy** to discriminate compact objects as a function of their “compactness” (or “closeness” to a black hole), “shadow” and gravitational waves
- Hypothetical binary ***SU*(*N*) gluons star** could be disentangled from a binary black hole system, due to possible differences in the gravitational wave frequency and amplitude and in the **mass-radius relation**
- No accompanying luminous signal is expected, unlike generic beyond SM theories equipped with electroweak interactions
- In extremely high pressure and temperature astrophysical phases, **unbroken *G*(2)** quarks can combine into multiquarks particles or can be seized by *G*(2) gluons, rearranging known QCD EOS into a phase of color-singlet *qGGG* hybrids and exotic hadrons.

Conclusions

- **Mathematical realism** has been the guide and criterion to go beyond the Standard Model: f**undamental microscopic forces might be manifestation of the conceivable algebras** that can be built via the Cayley-Dickson construction process
- **Naturalness and the WIMP miracle have been substituted with an algebraic conjecture**
- **The automorphism correspondence** higlights the **mismatch between *SU*(3) strong force and octonions**: the automorphism group is the exceptional group $G(2)$, which contains $SU(3)$, but it is not exhausted by $SU(3)$ itself. In their difference new physics lies, in the form of six additional massive bosons organized in composite states, potentially disconnected by Standard Model dynamics: **the exceptional-colored G(2) gluons**
- **When the Universe cooled down**, reaching a proper far beyond TeV energy scale at which ***G*(2) becomes broken**, usual $SU(3)$ QCD appeared, while an extra Higgs mechanism produced a secluded sector of cold bosonic states
- **A minimal additional particle content for a minimal SM extension**: a heavy scalar Higgs particle, responsible for a Higgs mechanism symmetrical w.r.t. the electroweak one, and a bunch of massive gluons
- **Sedenions** show an additional property: they still have $G(2)$ as a fundamental automorphism, but "tripled" by an S_3 factor, which **resembles the three fermion families of the Standard Model** and its S_3 −invariant extension
- $G(2)$ could guarantee peculiar manifestations in extreme astrophysical compact objects, such as **boson stars made of *G*(2) glueballs**, which **can populate the dark halos** and be observed in the future studying their gravitational waves and dynamics.

Thank you!



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